

Covering Segments with Unit Squares

Ankush Acharyya, Subhas C. Nandy, Supantha Pandit, and Sasanka Roy

Indian Statistical Institute, Kolkata, India.

{ankush_r,nandysc,sasanka}@isical.ac.in, pantha.pandit@gmail.com

Abstract

We study several variations of line segment covering problem with axis-parallel unit squares in \mathbb{R}^2 . A set S of n line segments is given. The objective is to find the minimum number of axis-parallel unit squares which cover at least one end-point of each segment. The variations depend on the orientation and length of the input segments. We prove some of these problems to be NP-complete, and give constant factor approximation algorithms for those problems. For some variations, we have polynomial time exact algorithms. For the general version of the problem, where the segments are of arbitrary length and orientation, and the squares are given as input, we propose a factor 16 approximation result based on multilevel linear programming relaxation technique, which may be useful for solving some other problems. Further, we show that our problems have connections with the problems studied by Arkin et al. [2] on conflict-free covering problem. Our NP-completeness results hold for more simplified types of objects than those of Arkin et al. [2].

Keywords: Segment cover, unit square, NP-hardness, linear programming, approximation algorithms, PTAS.

1 Introduction

Covering is a well-studied problem in computer science and has applications in diverse settings. Here a universe, and a collection of subsets of the universe are given as input. A minimum number of subsets need to be picked to cover the universe. The general version of this problem is NP-complete [9]. Many researchers studied different variants of this problem. In this paper, we study different interesting variations of line segment covering problem.

The motivation of studying this problem comes from its applications to network security [12]. Here, the objective is to check the connectivity or security of a physical network. A set of physical devices is deployed over a geographical area. These devices are communicated to each other through physical links. The objective is to check the security of the network by placing minimum number of devices which can sense all the links whose at least one end-point lies inside a desired geometrical object (circle/square) centered around it. This problem can be modelled as *line segment covering problem*, where the links can be interpreted as segments and the objects can be interpreted as unit squares. Note that, the links are considered as straight lines. In [12], several other applications are also stated.

Let $S = \{s_1, s_2, \dots, s_n\}$ be a set of line segments in \mathbb{R}^2 . We say that an axis-parallel square t **covers** a line segment $s \in S$ if t contains at least one **end-point** of s . In this paper, we deal with two classes of covering problem: i) **continuous**, and ii) **discrete**.

CONTINUOUS COVERING SEGMENTS BY UNIT SQUARES (CCSUS): Given a set S of segments in \mathbb{R}^2 , the goal is to find a set T of unit squares which covers all segments in S , and the cardinality of the set T is minimum among all possible sets of unit squares covering S .

DISCRETE COVERING SEGMENTS BY UNIT SQUARES (DCSUS): Along with the set S of segments, here we are given a set T of unit squares in \mathbb{R}^2 , and the goal is to find a subset $T' \subseteq T$ of minimum cardinality which can cover all the segments in S .

We study the following variations of covering problem for line segments which are classified depending upon their lengths and orientations.

CONTINUOUS COVERING

- **CCSUS-H1-US:** Horizontal unit segments inside a unit height strip.
- **CCSUS-H1:** Horizontal unit segments.
- **CCSUS-HV1:** Horizontal and vertical unit segments.
- **CCSUS-ARB:** Segments with arbitrary length and orientation.

DISCRETE COVERING

- **DCSUS-ARB:** Segments with arbitrary length and orientation.

We define some terminologies and definitions used in this paper. We use *segment* to denote a line segment, and *unit square* to denote an axis-parallel unit square. For a given non-vertical segment s , we define $l(s)$ and $r(s)$ to be the left and right end-points of s . For a vertical segment s , $l(s)$ and $r(s)$ are defined to be the end-points of s with highest and lowest y -coordinates respectively. The *center* of a square t is the point of intersection of its two diagonals. We use $t(a, b)$ to denote a square of side length b and whose center is at the point a . Further, we define *right-half*($t(a, b)$) to be the portion of $t(a, b)$ to the right of the vertical line passing through the point a .

Definition 1. Two segments in S are said to be *independent* if no unit square can cover both the segments. A subset $S' \subseteq S$ is said to be an *independent set* if every pair of segments in S' is independent. A subset $S' \subseteq S$ of segments is said to be *maximal independent set* if for any $s \in S \setminus S'$, $S' \cup \{s\}$ is not an independent set.

1.1 Connection with the paper of Arkin et al. [2]

We point out an interesting connection between this paper and the paper of Arkin et al. [2]. They studied a family of covering problem, called the *conflict-free covering*. Given a set P of n color classes, where each color class contains exactly two points, the goal is to find a set of conflict-free objects of minimum cardinality which covers at least one point from each color class. An object is said to be *conflict-free* if it contains at most one point from each color class. They looked at both discrete (where the covering objects are given as a part of the input) and continuous (where the covering objects can be placed anywhere in the plane) versions of conflict-free covering problem. When the points are on real line and the covering objects are intervals, both discrete and continuous versions of conflict-free covering problem are studied by them. Further, when same colored points are either horizontally or vertically separated by unit distance and the covering objects are unit squares, they studied the continuous version of conflict-free covering problem.

Instead of line segments, if we consider a given set of colored line segments where each segment is of different color and restrict each unit square to be conflict-free, then our definition of covering all the objects is equivalent to the conflict-free covering problem of Arkin et al. [2].

1.2 Known results

Arkin et al. [1, 2] showed that, both discrete and continuous versions of conflict-free covering problem are NP-complete where the points are on a real line and objects are intervals of arbitrary length. These results are also valid for covering arbitrary length segments on a line with unit intervals (see Appendix A.1). They provided factor 2 and factor 4 approximation algorithms for the continuous and discrete versions of conflict-free covering problem with arbitrary length intervals respectively. If points of same color class are either vertically or horizontally unit separated, then they proved that the continuous version of conflict-free covering problem with axis-parallel unit square is NP-complete and proposed a factor 6 approximation algorithm. Finally, they remarked the existence of a polynomial time dynamic programming based algorithm for the continuous version of conflict-free covering problem with unit intervals where the points are on a real line and each pair of same color points is unit separated. Recently, Kobylkin [12] studied the problem of covering the edges of a given straight line embedding of a planar graph by minimum number of unit disks, where an edge is said to be covered by a disk if any point on that edge lies inside that disk. They proved NP-completeness results for some special graphs. A similar study is made in [13], where a set of line segments is given, the objective is to cover these segments with minimum number of unit disks, where the covering of a segment by a disk is defined as in [12]. They studied both the discrete and continuous versions of the problem. For continuous version, they proposed a PTAS where the segments are non-intersecting. For discrete version, they showed that the problem is APX-hard.

1.3 Our contributions

In Section 2, we first propose an $O(n \log n)$ time greedy algorithm for *CCSUS-H1-US* problem. This is used to propose a factor 2 approximation result for the *CCSUS-H1* problem.

Arkin et al. [2] showed that continuous version of conflict-free covering problem of points by unit squares is NP-complete, where each pair of same color points is either horizontally or vertically unit distance apart. They also proposed a factor 6 approximation algorithm for this problem. We show that the *CCSUS-H1* problem is NP-complete. Thus our NP-completeness reduction works for more simplified types of objects than those of Arkin et al. [2]. In addition, we propose an $O(n \log n)$ time factor 3 approximation algorithm for the *CCSUS-HV1* problem. Finally, we provide a PTAS for *CCSUS-HV1* problem. We also give an $O(n \log n)$ time factor 6 approximation algorithm for the *CCSUS-ARB* problem.

In Section 3, we give a polynomial time factor 16 approximation algorithm for the *DCSUS-ARB* problem. It uses multiple levels of LP-relaxation, and finally an LP-based approximation algorithm. This method is of independent interest since it may be used to get approximation algorithm for some other problem. The running time of our algorithm fully depends on the running time of solving linear programs. Getting an algorithm with approximation factor better than 16 for the *DCSUS-ARB* problem remains an interesting open question.

2 Continuous covering

Here, the segments are given, and the objective is to place minimum number of unit squares for covering at least one end-point of all the segments.

2.1 CCSUS-H1-US problem

Below, we give an $O(n \log n)$ time greedy algorithm for the *CCSUS-H1-US* problem. Let S be a set of n horizontal unit segments inside a horizontal unit strip. Start with an empty set T' . Sort the segments in S from left to right with respect to their right end-points. Repeat the following steps until S becomes empty. Select the first segment $s \in S$ which is not yet covered by the

last added square in T' . Place a unit square t inside the strip aligning its left boundary at $r(s)$, and mark all the segments that are covered by t . Put t in T' . Finally, return the set T' as the output. Using standard analysis of covering points on real line by unit intervals (See Cormen et al. [5], Exercise 16.2-5), we can prove the following theorem.

Theorem 1. *The worst case time complexity of our algorithm for the CCSUS-H1-US problem is $O(n \log n)$.*

2.2 CCSUS-H1 problem

Here, we prove that *CCSUS-H1* is NP-complete. Next, we propose an $O(n \log n)$ time factor 2 approximation algorithm for this problem.

2.2.1 NP-completeness

The hardness of this problem is proved by a reduction from the rectilinear version of planar 3 SAT (*RPSAT(3)*) problem [11], which is known to be NP-complete.

RPSAT(3) [11]: Given a 3 SAT problem ϕ with n variables and m clauses, where the variables are positioned on a horizontal line and each clause containing 3 literals is formed with three vertical line segments and one horizontal line segments. Each clause is connected with its three variables either from above or from below such that two no two line segments corresponding to two different variables intersect. The objective is to find a satisfying assignment of ϕ . See Figure 1a for an instance of *RPSAT(3)* problem. Here the solid (resp. dotted) *vertical segment* attached to the horizontal line of a clause represents that the corresponding variable appears as a positive (resp. negative) literal in that clause.

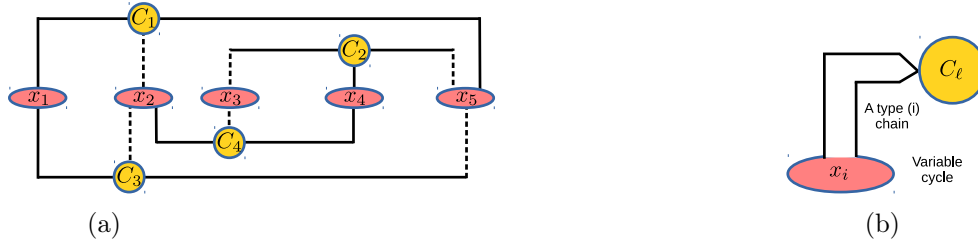


Figure 1: (a) *RPSAT(3)* representation. (b) Connection of a cycle and a chain.

We first describe the construction of an instance I of *CCSUS-H1* problem from an instance ϕ of *RPSAT(3)* problem. Next we validate the construction.

Let $\{x_1, x_2, \dots, x_n\}$ be n variables and $\{C_1, C_2, \dots, C_m\}$ be m clauses of ϕ . Here we describe the construction for the clauses connecting to the variables from above. A similar construction can be done for the clauses connecting to the variables from below.

Let d be the maximum number of vertical segments connected to a single variable from different clauses either from above or from below. Assume, $\delta = 4d + 3$. Each variable gadget for x_i may consist of a single *cycle* and at most $2d$ number of *chains*. The *cycle* consists of 2δ unit horizontal segments $\{s_1^i, s_2^i, \dots, s_{2\delta}^i\}$ in two sides of a horizontal line (see Figure 2). The segments $\{s_1^i, s_2^i, \dots, s_{\delta}^i\}$ are above the horizontal line and the segments $\{s_{\delta+1}^i, s_{\delta+2}^i, \dots, s_{2\delta}^i\}$ are below the horizontal line. The chains correspond to the vertical segments connecting a variable x_i with the clause containing it. There are three types of *chains*: (i) “ Γ ”, (ii) “ \neg ”, and (iii) “ \perp ” (see Figure 1a). The gadget corresponding to three types of chains are shown in Figures 3a, 3b, and 3c respectively. The chains are connected to the cycle, and together it forms a chain of *big-cycle* (see Figure 1b). It needs to mention that the number of segments is not fixed

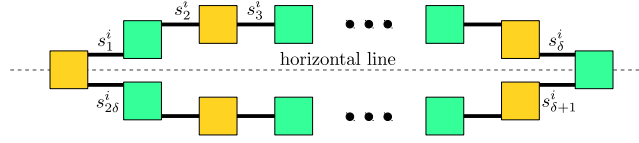


Figure 2: Gadget for a variable x_i .

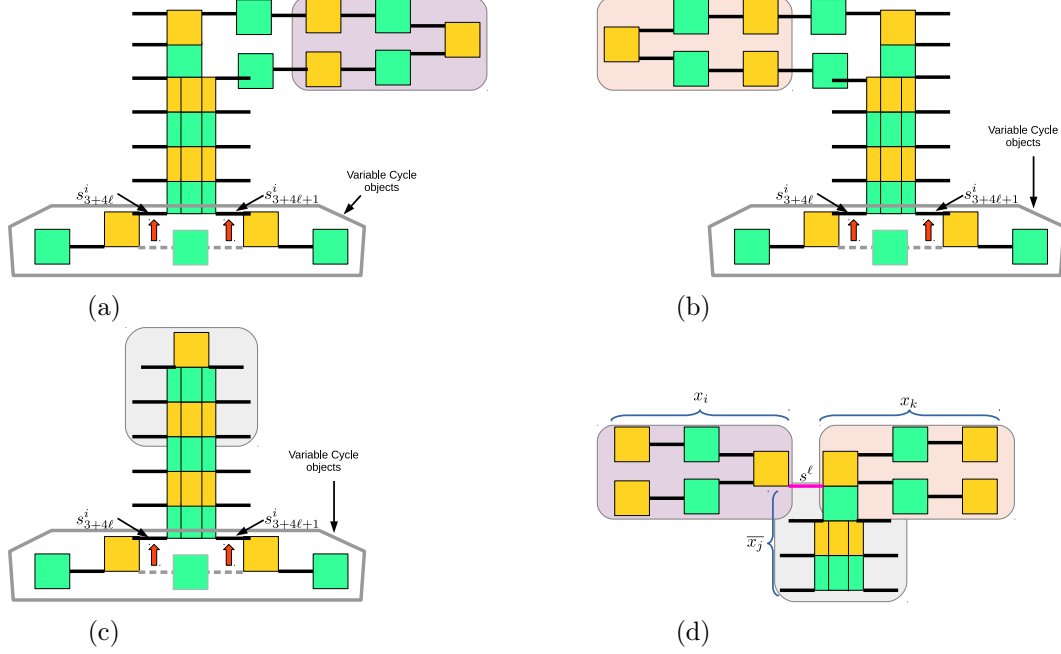


Figure 3: (a) Gadget for type (i) chain. (b) Gadget for type (ii) chain. (c) Gadget for type (iii) chain. (d) Demonstration of clause-segment s^ℓ corresponding to the clause $C_\ell = (x_i \vee \bar{x}_j \vee x_k)$; here the shaded portions from the parts (a), (b) and (c) are shown to demonstrate the connection of s^ℓ with the variables in C_ℓ .

for every chain, even for similar chains of different clauses. Note that, at the joining point (to construct a big-cycle) we slightly *perturb* two unit segments little upward.

Let $0, 1, 2, \dots, \kappa$ ($\kappa \leq d$) be the left to right order of the vertical segments corresponding to the clauses which are connected to the gadget corresponding to the variable x_i . Consider the ℓ -th clause C_ℓ in this order. If x_i is a positive literal, then the segments $s_{3+4\ell}^i$ and $s_{3+4\ell+1}^i$ are perturbed (moved upward as shown using upward arrow in Figures 3a, 3b, 3c) to connect the corresponding chain of C_ℓ with the cycle of variable x_i . Otherwise, If x_i is a negative literal, then the segments $s_{3+4\ell+1}^i$ and $s_{3+4\ell+2}^i$ are perturbed.

Note that, the squares are not given as a part of the input. In the Figures 2, 3a, 3b, and 3c a possible set of unit squares are also depicted. Each square can cover exactly two segments. Therefore, we have the following observation:

Observation 1. *Exactly half of the squares (either all green or all yellow) can cover all the segments in the big-cycle corresponding to the variable x_i . This solution represents the truth value (yellow for true and green for false) of the corresponding variable x_i .*

Further, for the clause C_ℓ , we take a single unit horizontal segment s^ℓ that connects the chain corresponding to three variables. This is referred to as a *clause-segment*. The placement of s^ℓ is shown in Figure 3d. Note that, in order to maintain the alternating green and yellow vertical layers in a variable gadget we may need to reduce the distance between two consecutive vertical

layers of squares. But, the segments are placed sufficiently apart so that no unit square can cover more than two segments from a variable gadget. As the number of segments (Q) considering all variable gadgets, is even, we need exactly $\frac{Q}{2}$ squares to cover them. Now, if a clause C_ℓ is *satisfiable* then at least one square connected to s^ℓ will be chosen, and hence s^ℓ will be covered; if C_ℓ is *not satisfiable* then the square adjacent to s^ℓ of each variable chain will not be chosen in the solution, and hence we need one more square to cover s^ℓ (see Figure 3d). Thus, we have the following result, which leads to Theorem 2.

Lemma 1. *The given RPSAT(3) formula is satisfiable if the number N of squares needed to cover all the unit segments in the construction is exactly $N_0 = \frac{1}{2}(\sum_{i=1}^n Q_i)$, where Q_i is the number of squares in the big-cycle corresponding to the gadget of the variable x_i . If the formula is not satisfiable then $N > N_0$,*

Theorem 2. *CCSUS-H1 is NP-complete.*

2.2.2 Approximation algorithm

Let S be a set of unit horizontal segments on the plane. We first partition the whole plane into a set of ℓ disjoint unit height horizontal strips H_1, H_2, \dots, H_ℓ . Let $S_i \in S$ be the set of segments in the strip H_i , for $i = 1, \dots, \ell$. Clearly, $S_i \cap S_j = \emptyset$, for $i \neq j$. Now we have the following observation.

Observation 2. *Any unit square cannot cover two segments, one from S_i and the other from S_j where $j - i \geq 2$, where $j > i$, for $i = 1, \dots, \ell - 2$, and $j = 3, \dots, \ell$.*

We calculate the minimum number of unit squares covering S_i (the segments in each H_i) using the algorithm in Section 2.1. Let Q_i be the set of squares returned for H_i in our algorithm. Let $Q^{odd} = \{Q_1 \cup Q_3 \cup \dots\}$ and $Q^{even} = \{Q_2 \cup Q_4 \cup \dots\}$ be the optimum solutions for the segments odd and even numbered strips respectively. We have $Q^{odd} \cap Q^{even} = \emptyset$, and we report $Q = Q^{odd} \cup Q^{even}$. Let OPT be a minimum sized set of unit squares covering S . Now, $|OPT| \geq \max(|Q^{odd}|, |Q^{even}|)$. Thus, $|Q| = |Q^{odd}| + |Q^{even}| \leq 2|OPT|$. Since $S_i \cap S_j = \emptyset$ and the time for computing Q_i is $O(|S_i| \log |S_i|)$ (by Theorem 1), the overall running time of the algorithm is $O(n \log n)$. Thus, we have the following theorem.

Theorem 3. *A 2-factor approximation result for the CCSUS-H1 problem can be computed in $O(n \log n)$ time.*

2.3 CCSUS-HV1 problem

Here, we have both horizontal and vertical segments in S which are of unit length. An easy way to get a factor 4 approximation algorithm for this problem is as follows. Let $S = S_H \cup S_V$, where S_H and S_V are the sets of horizontal and vertical unit segments respectively. We already have a factor 2 approximation algorithm for covering the members in S_H (see Theorem 3). The same algorithm works for S_V . Let Q_H and Q_V be the set of squares returned by our algorithm for covering S_H and S_V respectively. If OPT_H and OPT_V are the optimum solution for S_H and S_V respectively, and OPT be the overall optimum solution for $S_H \cup S_V$, then $|OPT| \geq |OPT_H|$ and $|OPT| \geq |OPT_V|$. Further, $|Q_H| \leq 2|OPT_H|$ and $|Q_V| \leq 2|OPT_V|$. Thus, $|Q_H| + |Q_V| \leq 2|OPT_H| + 2|OPT_V| \leq 4|OPT|$.

We now propose a factor 3 approximation algorithm for this problem using sweep-line technique. During the execution of the algorithm, we maintain a set of segments LB such that no two of the members in LB can be covered by an unit square. For each segment in S we maintain a



Figure 4: (a) Placement of 3 unit squares t_1, t_2, t_3 for a horizontal unit segment s . (b) Placement of 2 unit squares t_1 and t_2 for a vertical unit segment s .

flag variable; its value is 1 or 0 depending on whether it is covered or not by the chosen set of squares corresponding to the members in LB . We also maintain a range tree \mathcal{T} with the end-points of the members in S . Each element in \mathcal{T} has a pointer to the corresponding element in S . In Algorithm 1, we describe the algorithm.

Algorithm 1 Factor 3 Algorithm for *CHSUS-HV1*.

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1: Input: A set  $S$  of  $n$  horizontal and vertical unit segments.
2: Output: A set  $OUTPUT$  of unit squares which covers  $S$ .
3:  $OUTPUT = \emptyset$ ;  $LB = \emptyset$ 
4: sort the unit segments in  $S$  from top to bottom according to their  $r(\cdot)$ -values
5: (* see Section 1 for the definition of  $r(s)$ -values for the segments  $s \in S$  *)
6: for each segment  $s \in S$  in order do
7:   if  $flag(s) = 0$  then
8:     insert  $s$  in  $LB$ ; set  $flag(s) = 1$ 
9:     if  $s$  is horizontal then
10:       $m = 3$ , and define three unit squares  $\{t_1, t_2, t_3\}$  as shown in Figure 4a
11:      insert  $t_1, t_2, t_3$  in  $OUTPUT$ 
12:     else if  $s$  is vertical then
13:       $m = 2$ , and define two unit squares  $\{t_1, t_2\}$  as shown in Figure 4b
14:      insert  $t_1, t_2$  in  $OUTPUT$ 
15:     end if
16:     for  $i = 1, \dots, m$  do
17:       perform range searching with  $t_i$ 
18:       for each element  $\alpha$  of  $\mathcal{T}$  in  $t_i$  observe the corresponding element  $s' \in S$ 
19:       set  $flag(s') = 1$ ; delete both the end-points of  $s'$  from  $\mathcal{T}$ 
20:     end for
21:   end if
22: end for
23: Return  $OUTPUT$ 

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Theorem 4. *Algorithm 1 produces a 3-factor approximation result for the CHSUS-HV1 problem, and it runs in $O(n \log n)$ time using $O(n \log n)$ space.*

Proof. Let OPT be an optimal set of unit squares covering the members in S . In each iteration, we add a segment s to LB only if none of its end-points is covered by any unit square in $OUTPUT$. Clearly, LB is a maximal independent set of segments of S (see Definition 1) and hence $|LB| \leq |OPT|$. Further, for each segment added to LB , at most 3 unit squares are added to T' . Hence, $|T'| \leq 3|LB| \leq 3|OPT|$. Also, when the algorithm terminates the squares in $OUTPUT$ covers the segments S . By using range searching data structure Algorithm 1 can run in $O(n \log n)$ time. \square

2.3.1 Polynomial time approximation scheme

In this section, we propose a PTAS for the *CCSUS-HV1* problem using the *shifting strategy* of Hochbaum and Maass [8]. We are given a set S of n horizontal and vertical unit segments. Enclose the segments inside a integer length square box B ; partition B into vertical strips of width 1, and also partition B into horizontal strips of height 1. We choose a constant k , and define a *k -strip* which consists of at most k consecutive strips. Now, we define the concept of *shifts*. We have k different shifts in the vertical direction. Each vertical shift consists of some disjoint k -strips. In the i -th shift ($i = 0, 1, \dots, k-1$), the first k -strip consists of i unit vertical strips at extreme left, and then onwards each k -strip is formed with k consecutive unit vertical strips. Similarly, k shifts are defined in horizontal direction. Now consider *shift*(i, j) as the i -th vertical shift and j -th horizontal shift. This splits the box B into rectangular *cells* of size at most $k \times k$. The following observation is important to analyze the complexity of our algorithm.

Observation 3. *An optimal solution contains squares such that one boundary of each of those squares is attached to an end-point of some segment or two boundaries are attached to the end-point of two different segments.*

Justification: Suppose the boundary of a square t in the optimum solution does not pass through any end-point (see Figure 5a). We can move t vertically/horizontally to touch to an end-point of some segment (see Figure 5b). This square t can further be moved in the direction orthogonal to the previous movement to touch end-point of some other segment provided such an end-point exists within distance 1 from one of the boundaries of t in that direction (see Figure 5c). Figure 5d shows that a unit square in an optimum solution may touch an end-point of only one segment also.

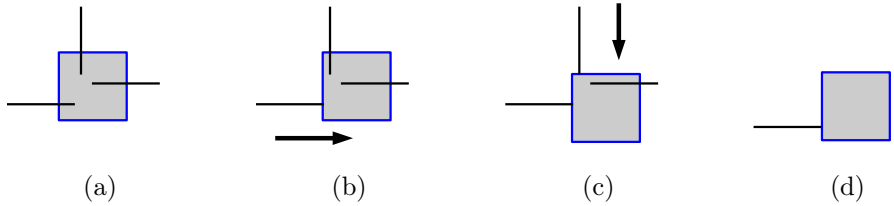


Figure 5: Justification of Observation 3.

Lemma 2. *Finding a feasible solution for each *shift*(i, j) require at most $O(n^{2k^2})$ time.*

Proof. First consider a single cell C of a particular *shift*(i, j) which consists of χ many 1×1 cell, where χ is at most k^2 . Let n_C be the size of the set of segments which has a portion inside C . Observe that, at most χ unit squares can cover these n_C segments in C . Again, at most $2n_C + 2 \times \binom{2n_C}{2}$ many positions are available for positioning the unit squares in an optimum solution of C (see Observation 3 and Figure 5)¹. We can use at most χ unit squares among $O(n_C^2)$ possible positions to cover all the segments in C . Since optimal solution may be of any size in $\{1, 2, \dots, \chi\}$, we may need to consider $O(n_C^{2\chi})$ possible configurations to get the optimal solution. Since each segment can participate in at most two cells in *shift*(i, j), we have $\sum_C n_C \leq 2n$. Thus, the time required for processing all the non-empty cells in *shift*(i, j) requires at most $O(n^{2k^2})$. \square

In our algorithm, for each *shift*(i, j) we calculate optimal solution in each cell and combine them to get a feasible solution. Finally, return the minimum among these k^2 feasible solutions.

¹See Appendix A.2 for the exact analysis of this count.

Let OPT be an optimum set of unit squares covering S , and Q be a feasible solution returned by our algorithm described above. Now, we prove the following theorem.

Theorem 5. $|Q| \leq (1 + \frac{1}{k})^2 |OPT|$ and the running time of the above algorithm is $O(k^2 n^{2k^2})$.

Proof. Let Q_{ij} be the solution of our algorithm for $shift(i, j)$. Also, assume that OPT_{ij} be the subset of squares in OPT such that each of them intersects the boundary of some cell in $shift(i, j)$. It can be shown that, $|Q_{ij}| \leq |OPT| + |OPT_{ij}|$ (Equation 2.3 of [8]). Now considering solutions for all $shift(i, j)$ for $1 \leq i, j \leq k$, we have

$$\sum_{i=1}^k \sum_{j=1}^k |Q_{ij}| \leq k^2 |OPT| + \sum_{i=1}^k \sum_{j=1}^k |OPT_{ij}|.$$

Each horizontal (resp. vertical) line may be considered at most k times during the k vertical (resp. horizontal) shifts. Thus, each square intersecting a horizontal (resp. vertical) line may be counted at most k times. Thus we have,

$$\sum_{i=1}^k \sum_{j=1}^k |OPT_{ij}| \leq k |OPT| + k |OPT|. \text{ (see [6] for a similar analysis)}$$

Hence, $\sum_{i=1}^k \sum_{j=1}^k |Q_{ij}| \leq (k^2 + 2k) |OPT| \leq (k + 1)^2 |OPT|$, and finally, we have

$$|Q| = \min_{i,j=1}^k \{|Q_{ij}|\} \leq \frac{\sum_{i=1}^k \sum_{j=1}^k |Q_{ij}|}{k^2} \leq \left(1 + \frac{1}{k}\right)^2 |OPT|.$$

Using Lemma 2, we conclude that the running time of our algorithm is $O(k^2 n^{2k^2})$. \square

2.4 CCSUS-ARB problem

Mimicking the factor 2 approximation algorithm for the vertex cover problem of a graph, we can have a factor 8 approximation algorithm for *CCSUS-ARB* problem as follows. Next, we improve the approximation factor to 6.

Observation 4. Let s_1 and s_2 be two segments in S . If none of the squares $t(l(s_1), 2)$ and $t(r(s_1), 2)$ covers s_2 , then s_1 and s_2 are independent².

As in Algorithm 1, here also we start with an empty set $OUTPUT$ and LB , and each segment in S is attached with a flag bit. We maintain a range tree \mathcal{T} with the end-points in S . Each time, an arbitrary segment $s \in S$ with $flag(s) = 0$ is chosen, and inserted in LB . Its flag bit is set to 1. Insert four unit squares $\{t_1, t_2, t_3, t_4\}$ which fully cover the square $t(l(s), 2)$ and four unit squares $\{t'_1, t'_2, t'_3, t'_4\}$ which fully cover the square $t(r(s), 2)$ in $OUTPUT$. Remove all the segments in S that are covered by $\{t_1, t_2, t_3, t_4, t'_1, t'_2, t'_3, t'_4\}$ by performing range searching in \mathcal{T} as stated in Algorithm 1. The end-points of the deleted segments are also deleted from \mathcal{T} . This process is repeated until all the members in S are flagged. Finally, return the set $OUTPUT$.

Theorem 6. The above algorithm for *CCSUS-ARB* problem runs in $O(n \log n)$ time, and produces a solution which is factor 8 approximation of the optimal solution.

Proof. The approximation factor follows from the fact that LB is a maximal independent set (see Observation 4), and for each member in LB we put 8 squares in $OUTPUT$. The time complexity follows from that of Algorithm 1. \square

²Note that, $t(a, b)$ is a $b \times b$ square with center at a .

We now improve the approximation factor to 6 using a sweep-line technique introduced in Biniat et al. [4]. We sort the segments in S with respect to their left end-points³, and process the elements in S in order. When an element $s \in S$ is processed, if $fls(s) = 0$ then we put six squares, two of them covering the 1×2 rectangle $t_1 = \text{right-half}(t(l(s), 2))$ and four of them covering the 2×2 square $t_2 = t(r(s), 2)$. We also identify the segments in S that are covered by t_1 and t_2 , and their flag-bit is set to 1. Thus, we have the following result:

Theorem 7. *The above algorithm for the CCSUS-ARB problem produces a factor 6 approximation result in $O(n \log n)$ time.*

Remark 1. *The above algorithm gives a factor 3 approximation result for the continuous covering horizontal segments of arbitrary length where the segments are inside a horizontal strip of unit height.*

3 Discrete covering: DCSUS-ARB problem

In this section, we give a 16 factor approximation algorithm for DCSUS-ARB problem. Let S be a set of n arbitrary segments and T be a set of m unit squares. The algorithm runs in a series of **steps**. In $(i + 1)$ -th step, we use linear programming to partition each subset of segments obtained in the i -th step into two disjoint subsets, and finally we obtain some subsets of S such that for each subset the objective is to cover either left or right end-points of all the segments with the portions of unit squares which are above/below a horizontal line. To prove the approximation factor, we consider the following problem.

COVERING POINTS BY UNIT WIDTH RECTANGLES ABUTTING x -AXIS (Restricted-Point-Cover): Given a set P of points in \mathbb{R}^2 and a set \mathcal{R} of unit width rectangles such that bottom boundary of each member in \mathcal{R} coincides with the x -axis, find a subset of \mathcal{R} of minimum cardinality to cover all the points in P .

Lemma 3. *If Z_{RPC} be the standard ILP formulation of the Restricted-Point-Cover problem, OPT_{RPC}^I and OPT_{RPC}^F are the optimum solutions of Z_{RPC} and its LP-relaxation respectively, then $OPT_{RPC}^I \leq 2OPT_{RPC}^F$.*

In Section 3.2 of [3], Bansal and Pruhs showed that $OPT_{RPC}^I \leq \alpha OPT_{RPC}^F$ for some positive constant α for a more generic version of this problem. We show that (in Appendix A.3) in our simplified case Lemma 3 follows.

Let Z_ν be an ILP. Denote \bar{Z}_ν , to be the LP-relaxation of Z_ν . Define OPT_ν^I and OPT_ν^F as the optimal solution of Z_ν and \bar{Z}_ν respectively. We first describe the different steps of the algorithm and finally establish the approximation factor.

Step 1: Let $T_1 \in T$ (resp. $T_2 \in T$) be the set of all squares which cover the left (resp. right) end-points of the segments in S . Now for each square $t_i \in T_1$, select a binary variable x_i , and for each square $t_j \in T_2$, select a binary variable y_j . Now create an ILP, Z_0 as follows.

$$\begin{aligned} Z_0 : \min \quad & \sum_{i|t_i \in T_1} x_i + \sum_{j|t_j \in T_2} y_j \\ \text{s.t.} \quad & \sum_{i|l(s_k) \in t_i \in T_1} x_i + \sum_{j|r(s_k) \in t_j \in T_2} y_j \geq 1 \quad \forall k \mid s_k \in S; \quad x_i, y_j \in \{0, 1\} \quad \forall i \mid t_i \in T_1 \text{ \& } j \mid t_j \in T_2 \end{aligned}$$

³The left end-point of a vertical segment is its top end-point.

After solving this *ILP*, the value of $x_i = 1$ or 0 depending on whether the square t_i is in an optimal solution or not. Similarly, $y_j = 1$ or 0 if the square t_j is in an optimal solution or not. We solve the corresponding *LP*, \bar{Z}_0 . Now, create two partitions $S_1 \subseteq S$ and $S_2 \subseteq S$ as follows. S_1 consists of those segments s_k such that $\sum_{i|l(s_k) \in t_i \in T_1} x_i \geq 1/2$, and S_2 consists of those segments s_ℓ for which $\sum_{j|l(s_\ell) \in t_j \in T_2} y_j \geq 1/2$. Now consider two *ILP*'s Z_1 and Z_2 as follows.

$$\begin{array}{l|l} Z_1 : \min \sum_{i|t_i \in T_1} x_i & Z_2 : \min \sum_{j|t_j \in T_2} y_j \\ \text{s.t.} \sum_{i|l(s_k) \in t_i \in T_1} x_i \geq 1, \quad \forall k \mid s_k \in S_1 & \text{s.t.} \sum_{j|r(s_k) \in t_j \in T_2} y_j \geq 1, \quad \forall k \mid s_k \in S_2 \\ x_i \in \{0, 1\} \quad \forall i \mid t_i \in T_1 & y_j \in \{0, 1\} \quad \forall j \mid t_j \in T_2 \end{array}$$

Then by an analysis identical to Gaur et al. [7], we conclude that $OPT_1^F + OPT_2^F \leq 2OPT_0^F \leq 2OPT_0^I$.

Step 2: Observe that, both the *ILP*'s, Z_1 and Z_2 are the problems of covering points by unit squares. Consider the covering problem Z_1 ⁴. Divide the plane into unit strips by drawing horizontal lines. Observe that, no unit square in T_1 can intersect more than one line. Partition T_1 into two sets T_{11} and T_{12} , where T_{11} consists of all squares which intersect even indexed lines and T_{12} consists of all squares which intersect odd indexed lines. Define binary variables x_i for $t_i \in T_{11}$ and y_j for $t_j \in T_{12}$. Then, Z_1 is equivalent to the following *ILP*.

$$\begin{array}{l} Z_1 : \min \sum_{i|t_i \in T_{11}} x_i + \sum_{j|t_j \in T_{12}} y_j \\ \text{s.t.} \sum_{i|l(s_k) \in t_i \in T_{11}} x_i + \sum_{j|l(s_k) \in t_j \in T_{12}} y_j \geq 1 \quad \forall k \mid s_k \in S_1; \quad x_i, y_j \in \{0, 1\} \quad \forall i \mid t_i \in T_{11} \text{ \& } j \mid t_j \in T_{12} \end{array}$$

We solve \bar{Z}_1 . Now, create two groups S_{11} and S_{12} . S_{11} consists of those segments s_k such that $\sum_{i|l(s_k) \in t_i \in T_{11}} x_i \geq 1/2$, and S_{12} consists of those segments s_ℓ for which $\sum_{j|l(s_\ell) \in t_j \in T_{12}} y_j \geq 1/2$. Again consider two *ILP*'s, Z_{11} and Z_{12} as follows.

$$\begin{array}{l|l} Z_{11} : \min \sum_{i|t_i \in T_{11}} x_i & Z_{12} : \min \sum_{j|t_j \in T_{12}} y_j \\ \text{s.t.} \sum_{i|l(s_k) \in t_i \in T_{11}} x_i \geq 1, \quad \forall k \mid s_k \in S_{11} & \text{s.t.} \sum_{j|l(s_k) \in t_j \in T_{12}} y_j \geq 1, \quad \forall k \mid s_k \in S_{12} \\ x_i \in \{0, 1\} \quad \forall i \mid t_i \in T_{11} & y_j \in \{0, 1\} \quad \forall j \mid t_j \in T_{12} \end{array}$$

Then by an analysis identical to Gaur et al. [7], we conclude that, $OPT_{11}^F + OPT_{12}^F \leq 2OPT_1^F$. A similar analysis for Z_2 leads to the following $OPT_{21}^F + OPT_{22}^F \leq 2OPT_2^F$.

Step 3: In step 2, there are four *ILP*'s, Z_{11} and Z_{12} corresponding to Z_1 and Z_{21} and Z_{22} corresponding to Z_2 . Observe that, Z_{11} is the problem of covering the left end-points of the segments $S_{11} \subseteq S_1$ by those unit squares which intersect even indexed lines. Similarly, Z_{12} is the problem of covering the left end-points of the segments $S_{12} \subseteq S_1$ by those unit squares which intersect odd indexed lines.

⁴More precisely, Z_1 (resp. Z_2) corresponds to the covering problem of left (resp. right) end points of the segments in S_1 (resp. S_2) by unit squares.

Now, focus our attention on Z_{11} . Since the squares intersected by an even indexed line ℓ_i is independent of the squares intersected by some other even indexed line ℓ_j and the points covered by the squares intersected by ℓ_i are not covered by the squares intersected by ℓ_j ($i \neq j$), we can split Z_{11} into different *ILPs* corresponding to each even indexed line, which can be independently solved.

Now consider Z_{11}^ξ corresponding to the line ℓ_ξ , where ξ is even. Let S_{11}^ξ be the set of segments whose left end-points are to be covered by the set of squares T_{11}^ξ , which are intersected by the line ℓ_ξ . We split S_{11}^ξ into disjoint sets S_{111}^ξ and S_{112}^ξ , where S_{111}^ξ (resp. S_{112}^ξ) are the set of all segments above (resp. below) the line ℓ_ξ . The objective is to cover the left end-points of the members S_{111}^ξ (resp. S_{112}^ξ) by minimum number of squares. Thus, the *ILP* of Z_{11}^ξ can be written as,

$$\begin{aligned} Z_{11}^\xi : \min \quad & \sum_{i|t_i \in T_{11}^\xi} x_i \\ \text{s.t.} \quad & \sum_{i|l(s_k) \in t_i \in T_{11}^\xi} x_i \geq 1 \quad \forall k \mid s_k \in S_{111}^\xi, \quad \text{and} \quad \sum_{i|l(s_k) \in t_i \in T_{11}^\xi} x_i \geq 1 \quad \forall k \mid s_k \in S_{112}^\xi \\ & x_i \in \{0, 1\} \quad \forall i \mid t_i \in T_{11}^\xi \end{aligned}$$

Again, since the sets S_{111}^ξ and S_{112}^ξ are disjoint, we may consider the following two *ILP*'s, Z_{111}^ξ and Z_{112}^ξ as follows.

$$\begin{array}{c|c} \begin{aligned} Z_{111}^\xi : \min \quad & \sum_{i|t_i \in T_{11}^\xi} x_i \\ \text{s.t.} \quad & \sum_{i|l(s_k) \in t_i \in T_{11}^\xi} x_i \geq 1 \quad \forall k \mid s_k \in S_{111}^\xi \\ & x_i \in \{0, 1\} \quad \forall i \mid t_i \in T_{11}^\xi \end{aligned} & \begin{aligned} Z_{112}^\xi : \min \quad & \sum_{i|t_i \in T_{11}^\xi} x_i \\ \text{s.t.} \quad & \sum_{i|l(s_k) \in t_i \in T_{11}^\xi} x_i \geq 1 \quad \forall k \mid s_k \in S_{112}^\xi \\ & x_i \in \{0, 1\} \quad \forall i \mid t_i \in T_{11}^\xi \end{aligned} \end{array}$$

Let \tilde{x}^* be an optimal fractional solution of \bar{Z}_{11}^ξ . Clearly, \tilde{x}^* satisfies all the constraints in both \bar{Z}_{111}^ξ and \bar{Z}_{112}^ξ . Also, it is observe that $OPT_{111}^{\xi F} \leq OPT_{11}^{\xi F}$ and $OPT_{112}^{\xi F} \leq OPT_{11}^{\xi F}$. Combining, we conclude that $OPT_{111}^{\xi F} + OPT_{112}^{\xi F} \leq 2OPT_{11}^{\xi F}$.

A similar equation can be shown for Z_{12}^ξ as follows: $OPT_{121}^{\xi F} + OPT_{122}^{\xi F} \leq 2OPT_{12}^{\xi F}$.

Finally, we have the following four equations,

1. $\sum_{\xi \text{ even}} OPT_{111}^{\xi F} + \sum_{\xi \text{ even}} OPT_{112}^{\xi F} \leq 2 \sum_{\xi \text{ even}} OPT_{11}^{\xi F} \leq 2OPT_{11}^F$,
2. $\sum_{\xi \text{ even}} OPT_{121}^{\xi F} + \sum_{\xi \text{ even}} OPT_{122}^{\xi F} \leq 2 \sum_{\xi \text{ even}} OPT_{12}^{\xi F} \leq 2OPT_{12}^F$,
3. $\sum_{\xi \text{ odd}} OPT_{211}^{\xi F} + \sum_{\xi \text{ odd}} OPT_{212}^{\xi F} \leq 2 \sum_{\xi \text{ odd}} OPT_{21}^{\xi F} \leq 2OPT_{21}^F$, and
4. $\sum_{\xi \text{ odd}} OPT_{221}^{\xi F} + \sum_{\xi \text{ odd}} OPT_{222}^{\xi F} \leq 2 \sum_{\xi \text{ odd}} OPT_{22}^{\xi F} \leq 2OPT_{22}^F$.

Step 4: In this step we apply Lemma 3 independently on each of Z_{111}^ξ , Z_{112}^ξ , Z_{121}^ξ , Z_{122}^ξ where ξ is even, and Z_{211}^ξ , Z_{212}^ξ , Z_{221}^ξ , Z_{222}^ξ where ξ is odd to get the following eight equations.

- | | | |
|---|--|---|
| (i) $OPT_{111}^{\xi I} \leq 2OPT_{111}^{\xi F}$, | (ii) $OPT_{112}^{\xi I} \leq 2OPT_{112}^{\xi F}$, | (iii) $OPT_{121}^{\xi I} \leq 2OPT_{121}^{\xi F}$, |
| (iv) $OPT_{122}^{\xi I} \leq 2OPT_{122}^{\xi F}$, | (v) $OPT_{211}^{\xi I} \leq 2OPT_{211}^{\xi F}$, | (vi) $OPT_{212}^{\xi I} \leq 2OPT_{212}^{\xi F}$, |
| (vii) $OPT_{221}^{\xi I} \leq 2OPT_{221}^{\xi F}$, | (viii) $OPT_{222}^{\xi I} \leq 2OPT_{222}^{\xi F}$. | |

Approximation factor: Now combining all the inequalities of Step 4, we have

$$\begin{aligned}
& \sum_{\xi \text{ even}} OPT_{111}^{\xi I} + \sum_{\xi \text{ even}} OPT_{112}^{\xi I} + \sum_{\xi \text{ even}} OPT_{121}^{\xi I} + \sum_{\xi \text{ even}} OPT_{122}^{\xi I} \\
& + \sum_{\xi \text{ odd}} OPT_{211}^{\xi I} + \sum_{\xi \text{ odd}} OPT_{212}^{\xi I} + \sum_{\xi \text{ odd}} OPT_{221}^{\xi I} + \sum_{\xi \text{ odd}} OPT_{222}^{\xi I} \\
& \leq 2(\sum_{\xi \text{ even}} OPT_{111}^{\xi F} + \sum_{\xi \text{ even}} OPT_{112}^{\xi F} + \sum_{\xi \text{ even}} OPT_{121}^{\xi F} + \sum_{\xi \text{ even}} OPT_{122}^{\xi F} \\
& + \sum_{\xi \text{ odd}} OPT_{211}^{\xi F} + \sum_{\xi \text{ odd}} OPT_{212}^{\xi F} + \sum_{\xi \text{ odd}} OPT_{221}^{\xi F} + \sum_{\xi \text{ odd}} OPT_{222}^{\xi F}) \\
& \leq 16 OPT_0^I, \text{ by applying the inequalities of Step 3, Step 2 and Step 1 in this order.}
\end{aligned}$$

Theorem 8. *There exists a factor 16 approximation algorithm for DCSUS-ARB problem that runs in polynomial time.*

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A Appendix

A.1 Conflict-free covering of points on a real line using unit intervals

Arkin et al [2] proved that the discrete version of conflict-free covering problem with intervals where the points are on real line is NP-complete, using a reduction from the vertex cover problem. We slightly modify this reduction as follows. Let $G(V, E)$ be a given graph with each vertex having distinct x -coordinate. Project each vertex $v \in V$ at a point x_v on the x -axis such that the distance between each pair of consecutive projections of vertices is strictly greater than 1. Now for each vertex $v \in V$, we take an unit interval I_v such that the mid point of I_v coincides with x_v on x -axis. Now, corresponding to each edge $e = (u, v)$, we take a single horizontal line segment connecting x_u and x_v . Thus we have an instance of the *DCSUS-ARB* problem where each interval represents a unit square with bottom boundary aligned to that interval. It is easy to show that any feasible solution of the *DCSUS-ARB* problem is the solution of the given vertex cover problem, and vice versa. Thus, we have the following observations.

Observation 5.

- *DCSUS-ARB problem is NP-complete even if the given segments are on a real line.*
- *Since the distance between any two consecutive projections (x_u, x_v) is greater than 1, an unit interval cannot cover end-points of segments on both x_u and x_v . Thus in order to cover the segments corresponding to the edges in G , we need to choose the unit squares (i.e., unit intervals) from I_v 's' as stated above. Thus, *CCSUS-ARB* is also NP-complete.*
- *Since vertex cover cannot be approximated better than factor 2 [10], both *CCSUS-ARB* and *DCSUS-ARB* cannot be approximated better than factor 2 even if the segments are on a real line.*

A.2 Analysis of the count on the available positions for placing unit squares in a cell of size $k \times k$

For a pair of end-points α and β of two different segments s_i and s_j respectively, $j \neq i$, with $d_\infty(\alpha, \beta)$ (L_∞ distance of α, β) ≤ 1 , we may have two distinct positions of placing an unit square. These squares will be referred to as *type-1* positions. However, if for an end-point α of some segment s_i , there is no end-point β of some other segment such that $d_\infty(\alpha, \beta) \leq 1$, then an unit square covering only s_i is required, and it can be placed anywhere containing an end-point of s_i . For the sake of simplicity we will take a unit-square whose top-left corner is anchored at such a point, and will refer such a position as *type-2* position. Thus, for each pair of segments s_i, s_j inside C , we may get at most two *type-1* positions, and for each segment s_i , we consider two *type-2* positions corresponding to the two end-points of s_i . Thus, the count stated in the proof of Lemma 2 follows.

A.3 Proof of Lemma 3

Consider the *ILP* formulation of *Restricted-Point-Cover* problem

$$\begin{aligned} Z_{\text{RPC}} : \quad & \min \sum_{r \in \mathcal{R}} x_r \\ \text{s.t.} \quad & \sum_{r|p \in r \text{ \& } r \in \mathcal{R}} x_r \geq 1, \quad \forall p \in P; \quad x_r \in \{0, 1\} \quad \forall r \in \mathcal{R} \end{aligned}$$

Let us consider the following algorithm. It uses two sets *LB* and *OUTPUT* as in Algorithm 1.

Procedure **RPC**(*P*, *R*)

1. Consider the point in *P* having maximum *y*-coordinate.
2. Identify the subset \mathcal{R}_p of rectangles in *R* that cover *p*.
3. Choose the $r_\ell \in \mathcal{R}_p$ having left-most left boundary, and $r_r \in \mathcal{R}_p$ having right-most right boundary.
4. Include $p \in LB$, and r_ℓ, r_r in *OUTPUT*.
Delete all the rectangles \mathcal{R}_p from *R*;
Delete the subset of points $P_p \subseteq P$ that are covered by the members in \mathcal{R}_p .
This splits (i) the remaining points into disjoint subsets P_1 and P_2 to the left of the left boundary of r_ℓ and to the right of the right boundary of r_r , and (ii) the remaining rectangles into disjoint subsets \mathcal{R}^1 and \mathcal{R}^2 that are to the left of the left boundary of r_ℓ and to the right of the right boundary of r_r .
5. If $P_1 \neq \emptyset$ then call **RPC**(P_1 , \mathcal{R}^1), and
If $P_2 \neq \emptyset$ then call **RPC**(P_2 , \mathcal{R}^2).
6. Report *OUTPUT*

Observe that, in the optimal solution of the *LP* \bar{Z}_{RPC} corresponding to the *ILP* Z_{RPC} , if $p \in LB$ then the constraint $\sum_{r \in \mathcal{R}_p} x_r \geq 1$ is satisfied. In other words, the sum of the variables corresponding to the rectangles in \mathcal{R}_p is at least 1. Also, these variables do not occur in the constraint for any other point $p' \in LB$. In order to satisfy the constraints corresponding to the points $p \in P \setminus LB$, some other variables may be assigned positive values. Thus, $OPT_{\text{RPC}}^F \geq LB$. In our solution of the *ILP* Z_{RPC} , we have chosen variables for setting value 1 such that the constraint corresponding to each $p \in LB$ attains the value 2. All other variables are set to 0. Thus, $OPT_{\text{RPC}}^I = 2LB$. Thus, the lemma is proved.